C GEOMETRIC SEQUENCES

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A sequence is geometric $\Leftrightarrow$ $\boldsymbol{\dfrac{u_{n+1}}{u_n} = r}$ for all $n \in \mathbb{Z}^+$.

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For example:

- 2, 10, 50, 250, ... is a geometric sequence as each term can be obtained by multiplying the previous term by 5.

  The common ratio is 5 since $\frac{10}{2} = \frac{50}{10} = \frac{250}{50} = 5$.
- 2, -10, 50, -250, ... is a geometric sequence with common ratio -5.

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The name "geometric" is given because the middle term of any three consecutive terms is the **geometric mean** of the terms on either side.

If the terms are a, b, c then $\dfrac{b}{a} = \dfrac{c}{b}$ {common ratio}

THE GENERAL TERM FORMULA

Suppose the first term of a geometric sequence is u_1 and the common ratio is r.

Then $u_2 = u_1 r, \quad u_3 = u_1 r^2, \quad u_4 = u_1 r^3,$ and so on.

Hence $u_n = u_1 r^{n-1}$

term number

The power of r is one less than the term number.

For a **geometric sequence** with **first term $\boldsymbol{u_1}$** and **common ratio $\boldsymbol{r}$**, the **general term** or **$\boldsymbol{n}$th term** is $\boldsymbol{u_n = u_1 r^{n-1}}$.

Example 7 **Self Tutor**

Consider the sequence $8, 4, 2, 1, \frac{1}{2}, ...$

a Show that the sequence is geometric.

b Find the general term u_n.

c Hence find the 12th term as a fraction.

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a Show that the sequence is geometric.

b Find the general term u_n.

c Hence find the 12th term as a fraction.

a $\frac{4}{8} = \frac{2}{4} = \frac{1}{2} = \frac{\frac{1}{2}}{1}$, so consecutive terms have the common ratio $\frac{1}{2}$.

$\therefore$ the sequence is geometric with $u_1 = 8$ and $r = \frac{1}{2}$.

b $u_n = u_1 r^{n-1}$

$\therefore \; u_n = 8\left(\frac{1}{2}\right)^{n-1}$ *or*

$$\begin{aligned} u_n &= 2^3 \times 2^{-(n-1)} \\ &= 2^3 \times 2^{1-n} \\ &= 2^{4-n} \end{aligned}$$

c

$$\begin{aligned} u_{12} &= 8 \times \left(\tfrac{1}{2}\right)^{11} \\ &= \tfrac{1}{256} \end{aligned}$$

1 For each of these geometric sequences:

i State u_1 and r.
ii Find the formula for the general term u_n.
iii Find the 9th term of the sequence.

a 3, 6, 12, 24, ...
b 2, 10, 50, ...
c 512, 256, 128, ...
d 1, 3, 9, 27, ...
e 12, 18, 27, ...
f $\frac{1}{16}, -\frac{1}{8}, \frac{1}{4}, -\frac{1}{2}, ...$

2 **a** Show that the sequence 5, 10, 20, 40, ... is geometric.
b Find u_n, and hence find the 15th term.

3 **a** Show that the sequence $12, -6, 3, -\frac{3}{2}, ...$ is geometric.
b Find u_n, and hence write the 13th term as a rational number.

4 **a** Show that the sequence $8, -6, 4.5, -3.375, ...$ is geometric.
b Hence find the 10th term as a decimal.

5 **a** Show that the sequence $8, 4\sqrt{2}, 4, 2\sqrt{2}, ...$ is geometric.
b Hence show that the general term of the sequence is $u_n = 2^{\frac{7}{2} - \frac{1}{2}n}$.

Example 8 **Self Tutor**

$k-1$, $2k$, and $21-k$ are consecutive terms of a geometric sequence. Find k.

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$k-1$, $2k$, and $21-k$ are consecutive terms of a geometric sequence. Find k.

Since the terms are geometric, $\frac{2k}{k-1} = \frac{21-k}{2k}$ {equating the common ratio r}

$$\begin{aligned}
\therefore\ 4k^2 &= (21-k)(k-1) \\
\therefore\ 4k^2 &= 21k - 21 - k^2 + k \\
\therefore\ 5k^2 - 22k + 21 &= 0 \\
\therefore\ (5k-7)(k-3) &= 0 \\
\therefore\ k &= \tfrac{7}{5} \text{ or } 3
\end{aligned}$$

Check: If $k = \frac{7}{5}$ the terms are: $\frac{2}{5}, \frac{14}{5}, \frac{98}{5}$. ✓ $\{r = 7\}$

If $k = 3$ the terms are: 2, 6, 18. ✓ $\{r = 3\}$

6 Find k given that the following are consecutive terms of a geometric sequence:

a $k,\ 3k,\ 54$

b $1000,\ 4k,\ k$

c $7,\ k,\ 28$

d $18,\ k,\ \frac{2}{9}$

e $k,\ 12,\ \dfrac{k}{9}$

f $k,\ 20,\ \frac{25}{4}k$

g $k,\ 3k,\ 20-k$

h $k,\ k+8,\ 9k$

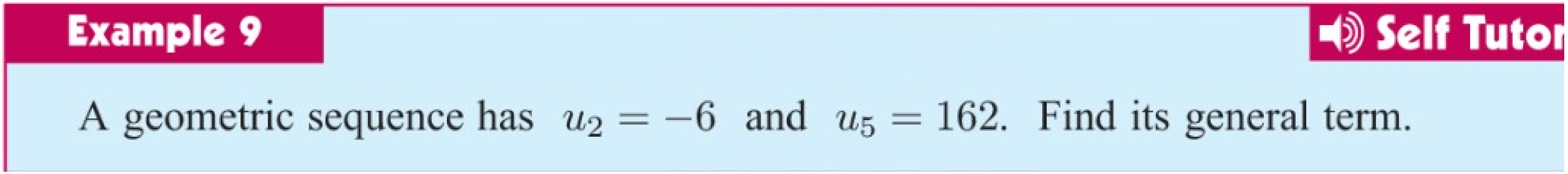
7 The first three terms of a geometric sequence are $k-1$, 6, and $3k$.

a Find the possible values of k.

b For each value of k, find the next term in the sequence.

Example 9 **Self Tutor**

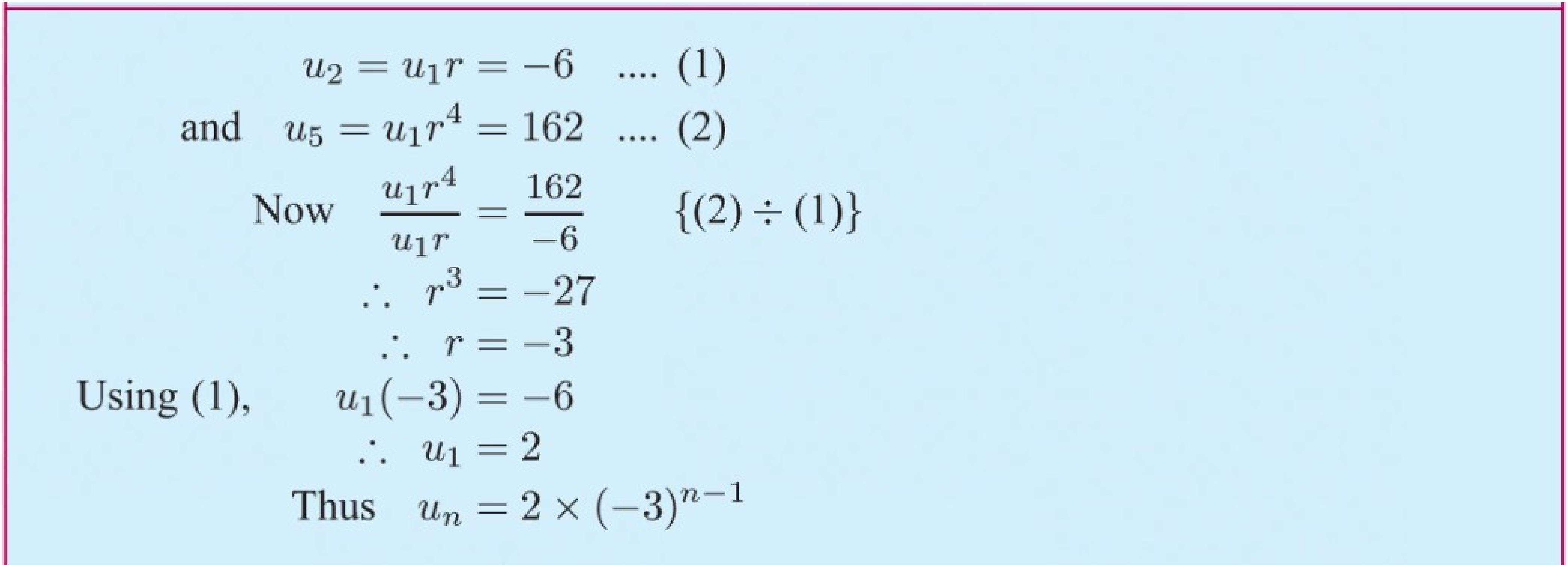
A geometric sequence has $u_2 = -6$ and $u_5 = 162$. Find its general term.



Example 9

Self Tutor

A geometric sequence has $u_2 = -6$ and $u_5 = 162$. Find its general term.



$$\begin{aligned} u_2 &= u_1 r = -6 \quad \text{....}\ (1) \\ \text{and} \quad u_5 &= u_1 r^4 = 162 \quad \text{....}\ (2) \\ \text{Now} \quad \frac{u_1 r^4}{u_1 r} &= \frac{162}{-6} \qquad \{(2) \div (1)\} \\ \therefore \quad r^3 &= -27 \\ \therefore \quad r &= -3 \\ \text{Using (1),} \quad u_1(-3) &= -6 \\ \therefore \quad u_1 &= 2 \\ \text{Thus} \quad u_n &= 2 \times (-3)^{n-1} \end{aligned}$$

8 Find the general term u_n of the geometric sequence which has:

a $u_4 = 24$ and $u_7 = 192$

b $u_3 = 8$ and $u_6 = -1$

c $u_7 = 24$ and $u_{15} = 384$

d $u_3 = 5$ and $u_7 = \frac{5}{4}$

9 A geometric sequence has $u_3 = 80$ and $u_6 = 270$.

a Find the first term and common ratio of the sequence.

b Find the 10th term of the sequence.

Example 10 **Self Tutor**

Find the first term of the sequence $6, 6\sqrt{2}, 12, 12\sqrt{2}, ...$ which exceeds 1400.

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Self Tutor

Find the first term of the sequence $6, 6\sqrt{2}, 12, 12\sqrt{2}, ...$ which exceeds 1400.

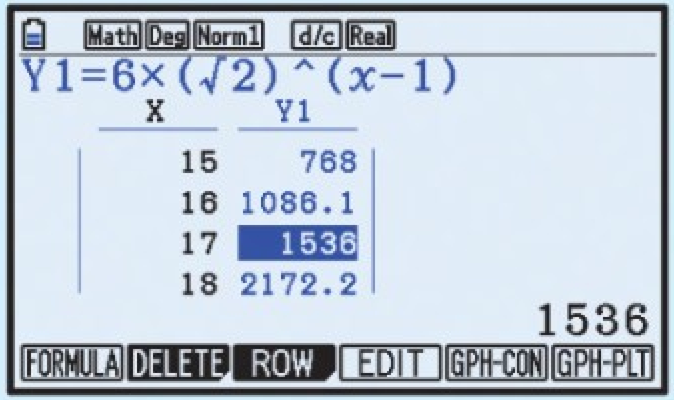
The sequence is geometric with $u_1 = 6$ and $r = \sqrt{2}$

$$\therefore \quad u_n = 6 \times (\sqrt{2})^{n-1}$$

We need to find n such that $u_n > 1400$.

Using a graphics calculator with $\text{Y}_1 = 6 \times (\sqrt{2})\hat{}(\text{X} - 1)$, we view a table of values:

GRAPHICS
CALCULATOR
INSTRUCTIONS



```
Y1=6×(√2)^(x−1)
       X        Y1
      15       768
      16    1086.1
      17      1536
      18    2172.2
                1536
```

The first term to exceed 1400 is $u_{17} = 1536$.

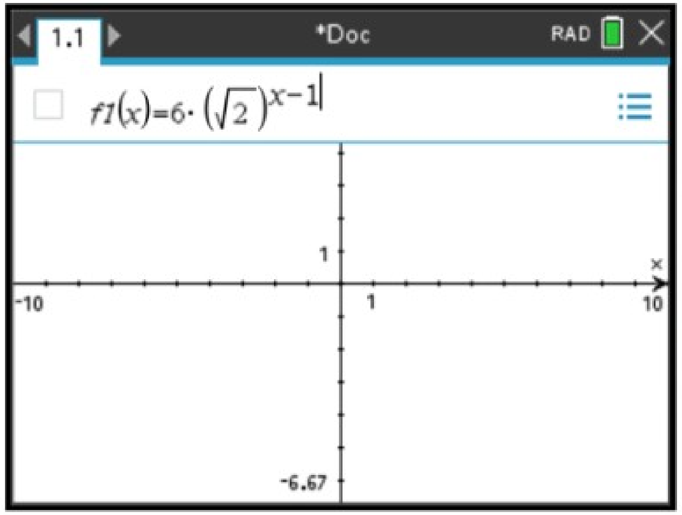
Find the first term of the sequence $6, 6\sqrt{2}, 12, 12\sqrt{2}, \ldots$ which exceeds 1400.

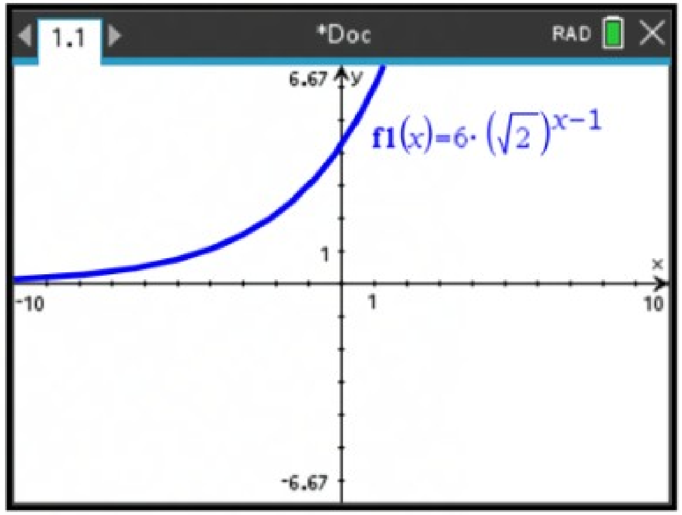
The sequence is geometric with $u_1 = 6$ and $r = \sqrt{2}$

$$\therefore\ u_n = 6 \times (\sqrt{2})^{n-1}$$

We need to find n such that $u_n > 1400$.

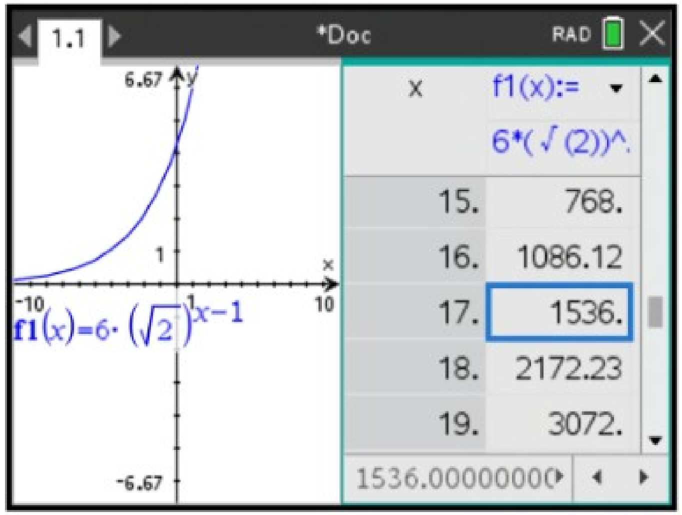
Press **ctrl** **doc ▼** (+ **page**), then select **2 Add Graphs** to open the Graphs application. Enter the formula for u_n into **f1(x)** as shown alongside, and press **enter**.





Press **ctrl** **T** to view a table of values for u_n, then scroll down to the first term that exceeds 1400.

The first term to exceed 1400 is $u_{17} = 1536$.



x	f1(x):= 6*(√(2))^.
15.	768.
16.	1086.12
17.	1536.
18.	2172.23
19.	3072.

1536.0000000

Add note...

Toggle notes

10 **a** Find the first term of the sequence $2, 6, 18, 54, ...$ which exceeds 10 000.

b Find the first term of the sequence $4, 4\sqrt{3}, 12, 12\sqrt{3}, ...$ which exceeds 4800.

c Find the first term of the sequence $12, 6, 3, 1.5, ...$ which is less than 0.0001.

d Find the first term of the sequence $5, -\frac{15}{2}, \frac{45}{4}, -\frac{135}{8}, ...$ which is less than -100.

11 What is the maximum number of distinct prime numbers that can occur in a geometric sequence? Give an example of such a sequence.

12 A geometric sequence with common ratio r and an arithmetic sequence with common difference d have the same first two terms. The third terms of the geometric and arithmetic sequences are in the ratio $2 : 1$.

a Find the possible values of r.

b For each value of r, find the ratio of the fourth terms of the sequences.

1 **a** **i** $u_1 = 3,\ r = 2$ **ii** $u_n = 3 \times 2^{n-1}$ **iii** $u_9 = 768$

b **i** $u_1 = 2,\ r = 5$ **ii** $u_n = 2 \times 5^{n-1}$

iii $u_9 = 781\,250$

c **i** $u_1 = 512,\ r = \frac{1}{2}$ **ii** $u_n = 512 \times 2^{1-n}$

iii $u_9 = 2$

d **i** $u_1 = 1,\ r = 3$ **ii** $u_n = 3^{n-1}$ **iii** $u_9 = 6561$

e **i** $u_1 = 12,\ r = \frac{3}{2}$ **ii** $u_n = 12 \times (\frac{3}{2})^{n-1}$

iii $u_9 = \dfrac{3^9}{2^6} = \dfrac{19\,683}{64}$

f **i** $u_1 = \frac{1}{16},\ r = -2$ **ii** $u_n = \frac{1}{16}(-2)^{n-1}$

iii $u_9 = 16$

2 **a** $u_1 = 5,\ r = 2$ **b** $u_n = 5 \times 2^{n-1},\ u_{15} = 81\,920$

3 **a** $u_1 = 12,\ r = -\frac{1}{2}$

b $u_n = 12 \times (-\frac{1}{2})^{n-1},\ u_{13} = \frac{3}{1024} \approx 0.002\,93$

4 **a** $u_1 = 8,\ r = -\frac{3}{4}$ **b** $u_{10} \approx -0.601$

5 **a** $u_1 = 8,\ r = \frac{1}{\sqrt{2}}$ **b** **Hint:** $u_n = 2^3 \times (2^{-\frac{1}{2}})^{n-1}$

6 **a** $k = 6$ **b** $k = \frac{125}{2}$ **c** $k = \pm 14$ **d** $k = \pm 2$

e $k = \pm 36$ **f** $k = \pm 8$ **g** $k = 2$ **h** $k = -2$ or 4

7 **a** $k = -3$ or 4 **b** For $k = -3$, next term is $\frac{27}{2}$.
For $k = 4$, next term is 24.

8 **a** $u_n = 3 \times 2^{n-1}$ **b** $u_n = 32 \times (-\frac{1}{2})^{n-1}$

c $u_n = 3 \times (\pm\sqrt{2})^{n-1}$ **d** $u_n = 10 \times (\pm\sqrt{2})^{1-n}$

9 **a** $u_1 = 35\frac{5}{9}$, $r = \frac{3}{2}$ **b** $u_{10} = 1366\frac{7}{8}$

10 **a** $u_9 = 13\,122$ **b** $u_{14} = 2916\sqrt{3} \approx 5050$

c $u_{18} = \frac{3}{32\,768} \approx 0.000\,091\,6$

d $u_{10} = -\frac{98\,415}{512} \approx -192$

11 2 primes
Consider $u_1 = 2$, $r = \frac{3}{2}$. The only other prime is $u_2 = 3$.

12 **a** $r = 2 \pm \sqrt{2}$

b For $r = 2 + \sqrt{2}$, geometric to arithmetic is $2 + 2\sqrt{2} : 1$.
For $r = 2 - \sqrt{2}$, geometric to arithmetic is $2 - 2\sqrt{2} : 1$.